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Introduction to "A Numerical Method for Solving Incompressible Viscous Flow Problems"

The following article is Chorin's first paper. It presents work that was part of his Ph.D. thesis at the Courant Institute of Mathematical Sciences at New York University. This is one of the earliest papers on a finite difference approximation to the incompressible Navier–Stokes equations in primitive variables

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\,\Delta\mathbf{u} \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1b}$$

where $\mathbf{u} = (u, v, w)$ is the velocity, p is the presssure, ρ_0 is the (constant) density, and ν is the kinematic viscosity. The principal difficulty in constructing such a finite difference method is in satisfying the continuity equation, or incompressibility constraint, (1b). There are several reasons why this term is troublesome. Attempts to enforce (1b) lead to a Poisson equation for the pressure p, the solution of which is far more time consuming than explicit discretizations of the hyperbolic and parabolic terms in (1a). Given the computers of the day, this was especially a barrier to three-dimensional computations. In addition, at the time it was difficult to determine the appropriate boundary conditions on p for this Poisson equation.

Most previous work on the numerical approximation of (1a)–(1b) was two dimensional and based on discretizing the stream function ψ and vorticity ω. These quantities are related by $\Delta\psi = -\omega$, in which case $\mathbf{u} = (\partial_y\psi, -\partial_y\psi)$ and (1b) is automatically satisfied (e.g., see [11]). This method requires the solution of a Poisson equation for the stream function ψ at every time step. A notable exception to the vorticity–stream function approach is the paper on the MAC method by Harlow and Welch [12]. In this paper, the authors present a staggered mesh formulation using primitive variables that is based on discretizing the pressure in such a way as to assure that $\nabla\cdot\mathbf{u}$ is zero at all points at the end of each time step. This leads to a Poisson equation for the pressure, which the authors solved using a line relaxation method.

In the "artificial compressibility method" Chorin presented an entirely new approach to satisfying the divergence free constraint (1b). Chorin's idea was to replace (1b) with the related hyperbolic problem

$$\delta\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} = 0, \tag{1b$'$}$$

where $p = \rho/\delta$, ρ is an "artificial density," and δ "is an artificial compressibility." The artificial compressibility δ is regarded as a computational parameter, which is related to an artificial sound speed c by

$$c^2 = \frac{1}{\delta}.$$

Replacing Eq. (1b) with (1b′) eliminates the Poisson problem for p and replaces it with the hyperbolic problem (1b′), thereby producing a method that is less computationally time consuming. This enabled Chorin to compute numerical approximations to solutions of the Navier–Stokes equations in three dimensions at a time when most other methods were limited to computing two-dimensional solutions.

In this article Chorin used the method of artificial compressibility to approximate solutions to the steady version of (1a)–(1b),

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\,\Delta\mathbf{u}, \tag{2a}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{2b}$$

In this case, if the numerical solution of (1a) and (1b′) converges to a (numerical) solution that is independent of time, then it will be a numerical approximation of (2a)–(2b). Viewed in this way, δ is a disposable parameter, like a relaxation parameter.

However, the method of artificial compressibility can also be used to approximate solutions of the time-dependent problem (1a)–(1b). Temam [17] has shown that under suitable hypotheses, for fixed δ, solutions $\mathbf{u}_\delta$ of the system

$$\frac{\partial\mathbf{u}_\delta}{\partial t} + (\mathbf{u}_\delta\cdot\nabla)\mathbf{u}_\delta + \frac{1}{2}(\nabla\cdot\mathbf{u}_\delta)\mathbf{u}_\delta = -\frac{1}{\rho_0}\nabla p + \nu\,\Delta\mathbf{u}_\delta, \tag{3a}$$

$$\delta\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u}_\delta = 0, \tag{3b}$$

exist and are unique and that these solutions converge to a solution $\mathbf{u}$ of (1a)–(1b) in the limit as $\delta \to 0$; i.e.,

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$$\mathbf{u}_\delta \to \mathbf{u} \quad \text{as} \quad \delta \to 0,$$

where $\mathbf{u}$ is a solution of (1a)–(1b). The term $1/2(\nabla \cdot \mathbf{u})\mathbf{u}$ in (3a) is a "stabilization term" that Temam needed in order to carry out his analysis. Bercovier and Engelman [1] developed a finite element "penalty" method for approximating solutions of (1a)–(2b) which is based on a finite element discretization of a modified version of (3a),

$$\frac{\partial \mathbf{u}_\delta}{\partial t} + (\mathbf{u}_\delta \cdot \nabla)\mathbf{u}_\delta + \frac{1}{2}(\nabla \cdot \mathbf{u}_\delta)\mathbf{u}_\delta + \frac{1}{\delta}\nabla(\nabla \cdot \mathbf{u}_\delta) = \nu\,\Delta \mathbf{u}_\delta. \quad (4)$$

In this approach the additional term $(1/\delta)\nabla(\nabla \cdot \mathbf{u}_\delta)$ is designed to penalize nonzero values of $\nabla \cdot \mathbf{u}_\delta$, in order to ensure that the condition $\nabla \cdot \mathbf{u}_\delta = 0$ is satisfied in the limit as $\delta \to 0$. One can also consider how the artificial compressibility method satisfies the constraint of incompressibility by extracting a penalty for every growth or decrease in the artificial density field. This has led to a variety of hybrid artificial compressibility/penalty methods.

After its introduction, the method of artificial compressibility was quickly adopted as one of the standard approaches for approximating solutions of (1a)–(1b) or (2a)–(2b). Chorin's original paper has been reprinted several times [5, 6] and several textbooks [9, 15, 17] have entire sections devoted to the method. It has been very widely used by researchers who wished to use existing software packages, originally developed for approximating solutions of compressible Euler or Navier–Stokes equations, to solve the incompressible version of these equations. This allowed workers to leverage the large amount of time invested in developing numerical methods to approximate solutions of the compressible Euler or Navier–Stokes equations—including grid generation and other ancillary software—to solve incompressible flow problems with few modifications of the original software.

This has spawned a substantial research effort, particularly in the computational aerodynamics community, to investigate problems associated with using the compressible Euler or Navier–Stokes equations to model low Mach number flow, i.e., problems for which

$$M \ll 1,$$

where $M^2 = (u^2 + v^2 + w^2)/a^2$ is the Mach number and $a = dp/d\rho$ is the sound speed. Representative examples of this work include Steger and Kutler [16], Choi and Merkle [4], Turkel [18] and Liou *et al.* [14]. An important outgrowth of this research is the work on "all speed" methods, i.e., methods that are designed to be effective at all Mach numbers (e.g., see Harlow and Amsden [10], or Chen and Pletcher [3]).

Today the method of artificial compressibility is the basis for a variety of numerical methods that are widely used by scientists and engineers in their work. It has been used to model a diverse range of problems, from the fluid mechanics associated with suspension feeding fishes [2] to the flow inside a thermal ink jet print head [8], and it is implemented as an option in at least one of the major commercial CFD packages (Flow 3D [13]).

We end with the observation that although Chorin's original work is primarily remembered for its contribution to the field of numerical methods for fluids, at the time he appeared to place more emphasis on the problem he wished to solve than on the numerical methodology he developed to solve it. Indeed, the first sentence in the abstract to his thesis [7] reads as follows: "The purpose of the present work is to study numerically the amplitude and the spatial structure of the laminar convective motions in a fluid layer heated from below." This emphasis on solving problems, rather than on developing methodology, can be seen in all of Chorin's subsequent work.

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